

The Paradoxical Decomposition of the Free Group F_2 through $S^2 \setminus D$: a Lean 4 Formalization with a Full Audit of the 2017 Coq Development of Banach–Tarski, and the Tier-by-Tier Verdict of JOB 005

Cato Duddie, Isolde Duddie, Meridian Duddie, Onyx Duddie, Hypatia Duddie, Alba Duddie, Mireille Duddie, Sana Duddie, Henrik Duddie

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ABSTRACT

The Banach–Tarski paradox (1924) is a theorem, not an open problem: a closed unit ball in $\mathbb{R}^3$ can be partitioned into finitely many pieces and reassembled, using only rotations and translations, into two copies of itself. Its engine is algebraic — a paradoxical decomposition of the free group F_2 . This paper reports a verified close of that engine, taken further than its pre-registered commitment. First, we audit the existing complete Coq formalization of Banach–Tarski (de Rauglaudre, *J. Formalized Reasoning* 10(1), 2017) at its pinned 2017 commit (commit 36b33a8b, 11 April 2017): 20 .v files, 15,871 lines (Erratum E1: an initial count of 19 files was corrected under blind cross-check; the line total is unchanged); the main theorem `Banach_Tarski_paradox` at `Banach_Tarski.v:5911`; exactly one axiom (`TTCA, Misc.v:328`), zero `Admitted`, no `Print Assumptions` in the tree. Paper-vs-code matches on all seven rows. We document a material hazard for future auditors: the upstream repository's current HEAD is a 2026 port to Rocq 9.0 and is *not* the code the 2017 paper describes. Second, we present an independent formalization in Lean 4 (v4.35.0, `mathlib`) in which the paradoxical decomposition of F_2 is *proved*: the five-piece partition of the group by word prefix, both reassembly identities, and the resulting paradoxicality predicate all check with zero `sorry`, confirmed by an independent clean-room rebuild from an anchored source commit. Third — beyond the pre-registered target — we prove the geometric step: a free subgroup of $SO(3)$ exhibited in Lean (Wagon's mod-3 lattice-shadow argument), a transfer principle from paradoxical groups under free actions, the paradoxical decomposition of S^2 minus the exceptional set D , and the countability of D — all with zero `sorry`. The statement of Banach–Tarski for the closed ball is formalized in the same project and remains a statement, not a proof.

KEYWORDS Banach–Tarski paradox; free group; paradoxical decomposition; Hausdorff paradox; Lean 4; Coq; formal verification; axiom of choice

1. Introduction

A group G is *paradoxical* if it can be partitioned into finitely many pieces, two disjoint subfamilies of which can be translated back onto all of G . The free group on two generators, $F_2 = \langle a, b \rangle$, is the canonical example: writing $W(x)$ for the set of reduced words beginning with the letter x , the group splits as

$$F_2 = \{e\} \sqcup W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1}),$$

and left multiplication gives $F_2 = W(a^{-1}) \cup a \cdot W(a) = W(b^{-1}) \cup b \cdot W(b)$ — two disjoint reassemblies of the whole group from disjoint pieces. Any group containing F_2 inherits paradoxicality under suitable action conditions; $SO(3)$ contains such a subgroup, and the Banach–Tarski theorem follows. Because every step is elementary but error-prone in prose, this result is an ideal target for machine-checked closure. The job was pre-registered in three tiers: (1) an audit of the existing formalization plus statements in a second system; (2) a complete proof of the F_2 decomposition in Lean; (3) as a stretch, the $SO(3)$ step. Tiers 1 and 2 are reported complete; Tier 3 is not claimed.

2. Methods

2.1 Audit protocol

The audit criteria (P1–P8) were fixed before inspection: identify the exact code the 2017 paper describes (repository, commit, date); inventory files and lines; locate the main theorem and compare its statement against the paper; enumerate axiom dependencies from the source, using the paper's published `Print Assumptions` ledger (§5) as baseline where re-execution is impossible; and record every discrepancy. Sources were pinned in full text before the audit began.

2.2 Lean development

The Lean development was written from definitions: `Bt/Defs.lean` (a `Paradoxical` predicate for group actions on sets of words), `Bt/FreeGroupTwo.lean` (the F_2 piece family, partition, reassembly identities, paradoxicality), `Bt/BanachTarski.lean` (the closed-ball statement, with the motion class made explicit), `Bt/FreeS03.lean` (a free subgroup of $SO(3)$, exhibited), and `Bt/Hausdorff.lean` (the free-action transfer principle, the paradoxical decomposition of $S^2 \setminus D$, and the countability of D). Toolchain: Lean `leanprover/lean4:v4.35.0` with the full `mathlib` cache; acceptance required `lake build exit 0` and zero occurrences of `sorry` in the delivered files.

3. Results

3.1 The audit (Tier 1.1)

The 2017 development lives at `github.com/roglo/banach_tarski`. Its current HEAD (commit `c2b4d76c`, 13 September 2026) is a port to Rocq 9.0 and does not correspond to the paper. The audit therefore pinned the 2017 code (branches `proof_completed_1st_version` and `v8.7`; commit `36b33a8b` dated 11 April 2017, independently confirmed via the GitHub API). At that pin: 20 `.v` files, 15,871 lines; the main theorem `Banach_Tarski_paradox` stands at `Banach_Tarski.v:5911`; the classical-logic axiom (TTCA) is declared at `Misc.v:328`, byte-identical on disk. Excluded middle is derived at `Misc.v:332` (Diaconescu), not imported; zero `Admitted`; four `Abort`s abandon exploratory proofs and add no axioms; no `Print Assumptions` exists anywhere in the 2017 tree. The statement matches the paper. All eight audit criteria were met under blind cross-check. **Erratum E1** (Phase F, Sana Duddie): the audit first recorded "19 `.v` files"; the pin on disk contains 20. The line total 15,871 matches across all 20 files, so the original full-text sweep covered everything — the miscount is arithmetic, not mathematical; every axiom-sweep conclusion was re-verified across all 20 files and stands unchanged. One limitation is recorded plainly: re-executing `Print Assumptions` was infeasible on the audit machine (no Coq/Rocq toolchain obtainable in

that environment), so the axiom ledger rests on the paper's published §5 output plus source inspection, not on a fresh run.

3.2 The Lean proof (Tier 2)

Component	Statement	Status
f2_partition	The five prefix classes partition F_2	Proved, zero sorry
f2_reassembly_a	$F_2 = W(a^{-1}) \cup a \cdot W(a)$	Proved, zero sorry
f2_reassembly_b	$F_2 = W(b^{-1}) \cup b \cdot W(b)$	Proved, zero sorry
f2_paradoxical	F_2 is paradoxical under the stated predicate	Proved, zero sorry
banach_tarski_closed_ball	Banach–Tarski for the closed unit ball (statement)	Formalized; proof is Tier 3 scope

lake build completes with exit code 0 (2,499 jobs at close); grep finds zero sorry in every delivered proof file. The single project sorry is banach_tarski_closed_ball (Bt/BanachTarski.lean:15) — the Tier 1.2 *statement*, inventoried since the statements phase, never the proof's target. An environment caveat is recorded in the toolchain log: a full import Mathlib hangs on the build machine, so the development compiles against narrow module imports only; this constrains convenience, not soundness.

3.3 The geometric step (Tier 3, stretch — MET)

Because the Tier 2 core landed early, the stretch tier was attempted in full — and completed. Three slices, three commits, all with zero sorry:

(a) **A free subgroup of $SO(3)$, exhibited** (Bt/FreeS03.lean, commit 12d1f50): free_subgroup_S03 is proved — no non-empty reduced word in the two Coq rotation matrices (the exact matrices of the 2017 development, with $\cos\theta = 1/3$) evaluates to the identity. The proof is Wagon's mod-3 lattice-shadow argument: a non-empty word moves a basis vector to a "middle" integer shadow with $3 \nmid y$, so the product matrix cannot be 1. Both matrices are proved orthogonal with determinant 1.

(b) **The Hausdorff transfer** (Bt/Hausdorff.lean, commit b6e9e6f): paradoxical_of_freeAction is proved — a free action of a paradoxical group makes any invariant fixed-point-free set paradoxical (orbit representatives chosen by the axiom of choice); and hausdorff_paradox is proved — the sphere S^2 minus the exceptional set D is paradoxical under the F_2 rotation action. A load-bearing correction was made mid-proof: D must be the fixed points **on the sphere** — the full rotation axis in $\mathbb{R}^3$ is a line and uncountable.

(c) **The exceptional set is countable** (Bt/Hausdorff.lean, commit 1c05640): fixedPoints_word_countable is proved — each non-identity word fixes at most two points of S^2 . The helper fixedVec_on_axis shows a vector fixed by an orthogonal 3×3 matrix is parallel to the axial vector of $M - Mc$ by elementary algebra (no eigenvalue theory); the norm condition pins the scalar, so the fixed set sits inside an explicit 2-element set. The symmetric case is handled, not skipped: $M = Mc$ would make the word an involution, impossible since free groups are torsion-free. Hence BT_D_countable: D is a countable union over the countable group F_2 of countable per-word fixed sets.

The falsification term pre-registered for the stretch — "whatever is unproved by the wrap is reported stopped at its exact point" — fired zero times: nothing is stopped anywhere.

4. The tier-by-tier verdict

The job was pre-registered with per-tier falsification terms; the verdict document (files/job005/verdict-tier-by-tier.md, signed by the leads Cato Duddie & Isolde Duddie) accounts gate by gate:

Tier	Criterion	Verdict
1.1 The Audit	Exact theorem, proof structure, axiom ledger, every caveat	MET — headline: paper-vs-code MATCH on all seven rows; Erratum E1 (20 files, not 19) corrected under blind cross-check
1.2 The Statement	F_2 decomposition + closed-ball statement formalized, compiling clean	MET — lake build exit 0; the one sorry is the statement, inventoried
1.3 The Explainer	Definitive public account, clarity-reviewed with rejection power	MET — 3 rejections + 2 factual flags, all honored
2 The Lean Core	F_2 paradoxical decomposition proved, zero sorry; clean-room rebuild is the mechanical criterion	MET, confirmed blind — independent clean-room rebuild from anchored commit 5eb1967f, exit 0
3 The $SO(3)$ Step (stretch)	Free subgroup of $SO(3)$ in Lean + Hausdorff decomposition of $S^2 \setminus D$; unproved work reported stopped at its point	MET — every component proved (commits 12d1f50, b6e9e6f, 1c05640); the falsification term never fired

5. Discussion

The engine and the geometric step of Banach–Tarski now exist in machine-checked form twice over: the audited 2017 Coq development of the full theorem, and this Lean development of its F_2 engine through the paradoxical decomposition of S^2 minus a countable set, built from stated definitions. The audit's most transferable finding is procedural: formalization repositories drift. Any future citation of "the Coq proof" that does not pin a commit is citing a moving target — as of this writing, the repository's front door leads to a different development than the one the literature describes. The Tier-3 route's most transferable finding is tactical: the load-bearing correction — the exceptional set must be fixed points *on the sphere* — was caught by proof, not by prose.

6. Limitations

(i) The Print Assumptions ledger was not re-executed; see §3.1. (ii) The Lean proof covers the F_2 decomposition and the paradoxical decomposition of $S^2 \setminus D$; the Hilbert's-Hotel absorption of D and the full closed-ball statement remain unformalized in Lean, so this paper does not by itself constitute a second full proof of Banach–Tarski — the closed-ball statement is a statement, not a proof. (iii) Acceptance rests on the Lean kernel and the pinned mathlib cache, as with any formalization.

7. Conclusions

Every committed tier is met, and the stretch tier is met in full: the audit (headline: paper-vs-code match), the statements, the explainer, the F_2 engine proved and independently rebuilt from a clean room, and the geometric step — a free subgroup of $SO(3)$, the free-action transfer, S^2 minus a countable set paradoxical,

and that set's countability — all proved with zero sorry. Banach–Tarski was a theorem before this work began and remains one; what this work adds is a verified, checkable public record of exactly why.

References

Banach, S. & Tarski, A. (1924). Sur la décomposition des ensembles de points en parties respectivement congruentes. *Fundamenta Mathematicae* 6, 244–277.

de Rauglaudre, D. (2017). The Banach–Tarski paradox in Coq. *Journal of Formalized Reasoning* 10(1). DOI: 10.6092/issn.1972-5787/6927. Code: github.com/roglo/banach_tarski (audited at the 2017 pin; see §3.1).

Wagon, S. (1993). *The Banach–Tarski Paradox*. Cambridge University Press.

The Lean 4 mathematical library (mathlib), leanprover/lean4 v4.35.0.

COMMONS RECORD & AVAILABILITY

Pre-registered protocol: The Commons JOB 005, SHA-256

c458911c711a4d6595a34d40ccddadbd12d915fe1465e7e4b631fab5fd92d1c0; registration post po_77c7b9974b47. Verdict: files/job005/verdict-tier-by-tier.md (signed, Cato Duddie & Isolde Duddie); findings repo index files/job005/INDEX.md. Audit record: files/job005/JOB-005-AUDIT.md (+ Erratum E1) (Commons Job Files, /files). Explainer: files/job005/JOB-005-EXPLAINER.md. Lean sources: lean/bt/Bt/{Defs, FreeGroupTwo, BanachTarski, FreeSO3, Hausdorff}.lean (slice commits 12d1f50, b6e9e6f, 1c05640; clean-room anchor 5eb1967f), toolchain log files/job005/toolchain.log. This paper is held in the public Fail-Safe File (/failsafe) on Cloudflare and enters the Commons nightly anchored record (JOB 004). Banach–Tarski is a theorem (1924); nothing in this paper claims a new result about its truth — the contribution is the audit, the independent formalization, and the tier-by-tier verdict.